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Hamiltonian dynamics on the symplectic extended phase space for autonomous and non-autonomous systems

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Abstract

We will present a consistent description of Hamiltonian dynamics on the 'symplectic extended phase space' that is analogous to that of a time-independent Hamiltonian system on the conventional symplectic phase space. The extended Hamiltonian H_1 and the pertaining extended symplectic structure that establish the proper canonical extension of a conventional Hamiltonian H will be derived from a generalized formulation of Hamilton's variational principle. The extended canonical transformation theory then naturally permits transformations that also map the time scales of the original and destination system, while preserving the extended Hamiltonian H_1, and hence the form of the canonical equations derived from H_1. The Lorentz transformation, as well as time scaling transformations in celestial mechanics, will be shown to represent particular canonical transformations in the symplectic extended phase space. Furthermore, the generalized canonical transformation approach allows us to directly map explicitly time-dependent Hamiltonians into time-independent ones. An 'extended' generating function that defines transformations of this kind will be presented for the time-dependent damped harmonic oscillator and for a general class of explicitly time-dependent potentials. In the appendix, we will re-establish the proper form of the extended Hamiltonian H_1 by means of a Legendre transformation of the extended Lagrangian L_1.

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1. Introduction

The modern description of time-independent Hamiltonian systems on symplectic manifolds is well established (e.g. Abraham and Marsden 1978, Arnold 1989, José and Saletan 1998, Marsden and Ratiu 1999, Frankel 2001). If the Hamiltonian H is explicitly time dependent, the carrier manifold of the Hamiltonian is of odd dimension, and the symplectic description is no longer appropriate. However, many features of the symplectic description can be extended to the 'presymplectic description' on an odd-dimensional contact manifold (Abraham and Marsden 1978, Marsden and Ratiu 1999, Frankel 2001). For instance, the theory of canonical transformations, hence mappings within a symplectic manifold that preserve its symplectic structure, can indeed be generalized on a presymplectic geometry. Nevertheless, the canonical transformation theory within the presymplectic context suffers from the restriction that the transformation must preserve time (Abraham and Marsden 1978, p 384). This means that both the original and the destination system are always correlated at the same instant of their respective time scales. Mappings, such as the Lorentz transformation, that necessitate a time shift between original and destination systems thus escape a description in terms of a canonical transformation within the presymplectic formalism. Furthermore, regularization transformations in celestial mechanics dating back to L Euler (Siegel and Moser 1971), as well as transformations of nonlinear, explicitly time-dependent Hamiltonian systems into time-independent systems (Struckmeier and Riedel 2001) are well known to require non-trivial mappings of the time scales.

Various approaches were made to describe these transformations within the context of a generalized canonical transformation theory (Lanczos 1949, Synge 1960, Szebehely 1967, Kuwabara 1984, Asorey *et al* 1983, Cariñena and Ibort 1987, Cariñena *et al* 1988). The underlying idea is to develop a generalized Hamiltonian formalism on a 'symplectic extended phase space' in analogy to the symplectic description on the conventional phase space of an autonomous Hamiltonian system ($\partial H/\partial t = 0$). Specifically, in the extended formalism, the time t is treated as an ordinary canonical function $t(s) \equiv q^{n+1}(s)$ of a new superordinated system evolution parameter, s. Its canonically conjugate function $p_{n+1}(s)$ will constitute the additional coordinate that renders the carrier manifold even-dimensional—and hence eligible for a symplectic description. The dynamics of the given system is then determined by an extended Hamiltonian H_1 with $\partial H_1/\partial s = 0$. As a consequence, all properties of Hamiltonian systems on symplectic manifolds can then be similarly reformulated within the extended Hamiltonian formalism. For example, time-dependent symmetries of explicitly time-dependent Hamiltonian systems can be treated like usual symmetries of autonomous Hamiltonian systems. Moreover, it is possible to define canonical transformations within the symplectic extended phase space that are more general than those within the lower dimensional presymplectic description. This formalism will then naturally permit generating functions of extended canonical transformations that also define a non-trivial mapping of the time scales of the original and the destination systems. Of course, an analogy with the conventional canonical transformation theory will require the extended Hamiltonian H_1 to be preserved under extended canonical transformations, and hence the form of the transformed canonical equations derived from H_1.

Following the pioneering works of Lanczos (1949, p 189) and Synge (1960, p 143), the extended Hamiltonian H_{LS} is commonly defined as the energy surface $H_{\mathrm{LS}} = H - \mathcal{H} = 0$, with $\mathcal{H}$ denoting the value of the conventional Hamiltonian H (e.g. Stiefel and Scheifele 1971, Thirring 1977, Asorey *et al* 1983, Kuwabara 1984, Lichtenberg and Lieberman 1992, Stump 1998, Tsiganov 2000, Struckmeier and Riedel 2002a). Yet, the so-defined extended Hamiltonian H_{LS} fails to meet the requirement of being preserved under extended canonical transformations that define a non-trivial time mapping $t(s) \mapsto t'(s)$. As a consequence, the Hamiltonian H_{LS} does not preserve the form of the canonical equations under non-trivial time transformations, but satisfies only the weaker condition of preserving the canonical form of the Hamilton–Jacobi equations (Synge 1960, Tsiganov 2000). With a missing canonical transformation rule for H_{LS}, the extended canonical transformation formalism based on H_{LS} is *incomplete*.

In order to consistently construct an extended Hamiltonian on the symplectic extended phase space, other approaches (e.g. Gotay 1982, Cariñena and Ibort 1987, Cariñena *et al* 1988) pursued the idea of a 'coisotropic embedding' of the presymplectic geometry of a time-dependent H into the geometry of the symplectic extended phase space as the carrier manifold of the extended Hamiltonian. This geometric reasoning led to a rather general form of an extended Hamiltonian H_C that is not necessarily physical. Specifically, the proposed extended Hamiltonian $H_C = f(H - \mathcal{H})$, with f an arbitrary function of the canonical variables and time (Cariñena and Ibort 1987, Cariñena *et al* 1988), is not compatible with Hamilton's variational principle and does not yield an analogous description of the system's dynamics.

So, despite the fact that the idea of a generalized formulation of Hamiltonian dynamics on an extended symplectic manifold is long established, we can state that a consistent formulation that is analogous to the conventional symplectic description has not yet been worked out. With this paper, we aim to provide a consistent symplectic theory of the extended phase space that is based on a canonically invariant extended Hamiltonian function H_1. The derivation of the extended Hamiltonian H_1 from a generalized formulation of Hamilton's principle will, therefore, be the starting point of our analysis in section 2.1. It will turn out that our extended Hamiltonian $H_1 = k(H - \mathcal{H})$, with $k = \mathrm{d}t/\mathrm{d}s$ coincides with the Hamiltonian of the well-known Poincaré time transformation (Siegel and Moser 1971, p 35), also referred to as the symplectic time rescaling method in the realm of molecular dynamics. In conjunction with the extended symplectic 2-form, we will show in section 2.2 that it is exactly this extended Hamiltonian H_1 that permits a description of Hamiltonian dynamics on the symplectic extended phase space that is completely analogous to the conventional symplectic description of time-independent Hamiltonian systems on the non-extended, conventional phase space.

In section 2.3, we will derive a consistent formalism of extended canonical transformations within the symplectic extended phase space that preserve the extended Hamiltonian H_1 and, therefore, the form of the canonical equations. In this description, the evolution parameter s serves as the independent variable that is common to both the original and the destination system. In this respect, the new evolution parameter s plays exactly the role of the time t in the conventional canonical transformation theory. The generalized formulation of Hamilton's variational principle thus establishes the basis for the definition of *extended generating functions* for *finite* canonical transformations that make it possible to relate both systems at *different* instants of their respective time scales, $t(s)$ and $t'(s)$. We will furthermore formulate the extended version of Liouville's theorem that applies to the symplectic extended phase space. In addition, the restrictions will be worked out that are to be imposed on the functional structure of extended generating functions in order for the transformed time $t'(s)$ to sustain the meaning of $t(s)$ as a common parameter for all canonical variables p'_i and q'^i.

As a first example of an extended canonical transformation, we will show in section 3.1 that the Lorentz transformation can be formulated as a particular canonical transformation in the extended phase space. Since its generating function does not explicitly depend on s, we will encounter the interesting result that the extended Hamiltonian H_1 is Lorentz-invariant.

In celestial mechanics, regularization transformations of many-body systems are known to require a replacement of the physical time t by a 'fictitious' time t'. We will show in section 3.2 that these transformations can be formulated as *finite* canonical transformations in the symplectic extended phase space that preserve the extended Hamiltonian H_1. We thereby integrate the useful and well-established regularization techniques of celestial mechanics into the framework of a now consistent generalized formulation of the canonical transformation theory.

In section 3.3, an extended generating function will be presented that defines a canonical mapping of a general class of explicitly time-dependent Hamiltonian systems into time-independent ones. Demanding the transformed system to be autonomous then determines the time correlation of both systems. For the simple but important case of a time-dependent damped harmonic oscillator, the extended canonical transformation yields the well-known invariant given by Leach (1978). For the general class of nonlinear and explicitly time-dependent Hamiltonian systems treated in section 3.4, we will show that the canonical transformation establishes a *linear mapping* of the system's global quantities energy $\mathcal{H}(t)$ and second moments $\boldsymbol{q}(t)\boldsymbol{p}(t)$ and $\boldsymbol{q}^2(t)$ into their respective initial values $\mathcal{H}_0$, $\boldsymbol{q}_0\boldsymbol{p}_0$ and $\boldsymbol{q}_0^2$.

In the appendix, the concept of a parameterization of time will be reviewed for Lagrange's formulation of dynamics. By means of a Legendre transformation of the extended Lagrangian L_1, we will re-establish our extended Hamiltonian H_1 of section 2.1. We thereby confirm that it is precisely this extended Hamiltonian that is compatible with the extended formulation of Hamilton's variational principle, and hence with a consistent formulation of Hamiltonian dynamics on the symplectic extended phase space.

2. Hamiltonian formalism in the symplectic extended phase space

2.1. Hamilton's variational principle, extended Hamiltonian

We consider an explicitly time-dependent Hamiltonian H that is defined on a finite-dimensional contact manifold $T^*Q \times \mathbb{R}$ with its closed, generally degenerate contact 2-form $\omega_H = \omega - \mathrm{d}H \wedge \mathrm{d}t$. Herein, ω stands for a symplectic, i.e. closed, non-degenerate and antisymmetric 2-form that renders the manifold T^*Q symplectic. On an exact symplectic manifold, there exists a 1-form λ with exterior derivative $\mathrm{d}\lambda = \omega$. Hamilton's variational principle states that the actual system trajectory $C_0 \subset T^*Q \times \mathbb{R}$ is the critical point of a path map $\mathfrak{S} : C \to \mathbb{R}$ with $\mathrm{d}\mathfrak{S}(C_0) = 0$. The map $\mathfrak{S}$ is defined as the integral along a path $C \subset T^*Q \times \mathbb{R}$ over the contact 1-form $\lambda - H\,\mathrm{d}t$, hence

$$\mathfrak{S}(C) = \int_C (\lambda - H\,\mathrm{d}t), \qquad \mathrm{d}\mathfrak{S}(C_0) = 0. \tag{1}$$

In canonical coordinates, we have $\lambda = \boldsymbol{p}\,\mathrm{d}\boldsymbol{q}$, with $(\boldsymbol{q}, \boldsymbol{p}) \in T^*Q$ the pair of n-component vectors and covectors. The actual variation of C is performed on the presymplectic manifold $T^*Q \times \mathbb{R}$. As the basis for a symplectic description, we reformulate equation (1) by treating the time $t = t(s) \equiv q^{n+1}(s)$ as an ordinary canonical variable that now depends, like all other canonical variables, on a newly introduced superordinated system evolution parameter s. To this end, we first introduce formally the extended configuration manifold as the product manifold $Q_1 := Q \times \mathbb{R}$, whose elements, in coordinate representation, comprise the vectors $\boldsymbol{q}_1 \equiv (\boldsymbol{q}, t) \in Q_1$. The extended Hamiltonian H_1 is then to be defined as a differentiable function on the cotangent bundle T^*Q_1 as the carrier manifold. Following the usual nomenclature of T^*Q as the 'phase space', we refer to the symplectic manifold T^*Q_1 as the 'symplectic extended phase space'. With respect to a canonical basis of a chart $U_1 \subset T^*Q_1$ and $\boldsymbol{p}_1 \equiv (\boldsymbol{p}, p_{n+1}) \in T^*_{\boldsymbol{q}_1}Q_1$, the extended Hamiltonian $H_1(\boldsymbol{q}_1, \boldsymbol{p}_1)$ thus maps all pairs of $(n+1)$-component vectors and covectors $(\boldsymbol{q}_1, \boldsymbol{p}_1) \in U_1$ into $\mathbb{R}$. Of course, T^*Q_1 embodies a cotangent bundle, hence a symplectic manifold, if and only if the symplectic structure ω on T^*Q can be 'extended' to a symplectic, i.e. non-degenerate structure Ω on T^*Q_1. This is achieved by a proper choice of p_{n+1}.

With $\mathfrak{S}_1 : \tilde{C} \to \mathbb{R}$ denoting a mapping of paths $\tilde{C} \subset T^*Q_1$ into $\mathbb{R}$, Hamilton's variational principle of equation (1) can be written equivalently in terms of the extended 1-form $\Lambda = \lambda + p_{n+1}\,\mathrm{d}t$ as the integral

$$\mathfrak{S}_1(\tilde{C}) = \int_{\tilde{C}} (\Lambda - H_1\,\mathrm{d}s), \qquad \mathrm{d}\mathfrak{S}_1(\tilde{C}_0) = 0, \tag{2}$$

with

$$H_1\,\mathrm{d}s = (H + p_{n+1})\,\mathrm{d}t. \tag{3}$$

In canonical coordinates, the extended 1-form is given by $\Lambda = \boldsymbol{p}_1\,\mathrm{d}\boldsymbol{q}_1$. The variation of the action integral (2) is now to be performed by varying $\tilde{C} \subset T^*Q_1$. We will see later that the critical path $\tilde{C}_0$ is compatible with the path C_0 following from (1).

In the case of an autonomous system, the Hamiltonian $H = \text{const}$ defines a $(2n - 1)$-dimensional hypersurface in T^*Q. The corresponding requirement for H_1—considering that the Hamiltonian is only determined up to an arbitrary additive constant—then suggests to define H_1 as an implicit function $H_1 = 0$, which then defines a $(2n + 1)$-dimensional hypersurface in T^*Q_1. With regard to equation (3), this, in turn, implies to define $-p_{n+1}$ as the *value* $\mathcal{H}$ of the system's Hamiltonian H

$$-p_{n+1}(s) \equiv \mathcal{H}(s) = H(\boldsymbol{q}(s), \boldsymbol{p}(s), t(s)). \tag{4}$$

The notation $\mathcal{H}$ is used to distinguish the Hamiltonian *function* H, defined on $T^*Q \times \mathbb{R}$, from its *value* $\mathcal{H}(s) \in \mathbb{R}$ as a new s-dependent canonical variable. Provided that the Hamiltonian represents the sum of the system's kinetic and potential energies, $\mathcal{H}(s)$ quantifies the system's instantaneous energy content. The negative system energy content—$\mathcal{H}$ thus embodies the canonical variable that is conjugate to the canonical variable time t. As will be shown in appendix A.2, the result $p_{n+1}(s) = -\mathcal{H}(s)$ follows also from the derivative of the extended Lagrangian L_1 with respect to the fibre $\mathrm{d}t/\mathrm{d}s$.

In canonical coordinates, the transition from the presymplectic carrier manifold of H to the symplectic extended phase space is thus given by the map $\psi : T^*Q \times \mathbb{R} \to T^*Q_1$

$$(\boldsymbol{q}, \boldsymbol{p}, t) \overset{\psi}{\mapsto} (\boldsymbol{q}, \boldsymbol{p}, t, \mathcal{H}), \qquad \mathcal{H} = H(\boldsymbol{q}, \boldsymbol{p}, t).$$

The inverse mapping $\psi^{-1} : T^*Q_1 \to T^*Q \times \mathbb{R}$ thus replaces the $\mathcal{H}$-terms by the Hamiltonian function H. The general, coordinate-free equation that uniquely relates a given Hamiltonian $H : T^*Q \times \mathbb{R} \to \mathbb{R}$, to its extension $H_1 : T^*Q_1 \to \mathbb{R}$ is thus obtained from (3) as the equation

$$H_1\,\mathrm{d}s = (H - \mathcal{H})\,\mathrm{d}t. \tag{5}$$

We emphasize that H_1 is *uniquely determined* by the setting $p_{n+1} = -\mathcal{H}$ and by the requirement that the conventional form of Hamilton's variational principle from equation (1) be equivalent to its extended formulation in equation (2).

With the extended covector $\boldsymbol{p}_1 \equiv (\boldsymbol{p}, -\mathcal{H})$, the canonical coordinate representation of the extended Hamiltonian H_1 is given by

$$H_1(\boldsymbol{q}_1, \boldsymbol{p}_1) = k[H(\boldsymbol{q}, \boldsymbol{p}, t) - \mathcal{H}], \qquad k = \frac{\mathrm{d}t}{\mathrm{d}s}. \tag{6}$$

The canonical representation of the extended symplectic structure $\Omega = \mathrm{d}\Lambda$ on T^*Q_1 is then with the additional pair $(q^{n+1}, p_{n+1}) \equiv (t, -\mathcal{H})$ of canonical coordinates

$$\Omega = \sum_{i=1}^{n+1} \mathrm{d}p_i \wedge \mathrm{d}q^i = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t. \tag{7}$$

Remarkably, the extended Hamiltonian H_1 in the form of equation (6) was first introduced by Poincaré (Siegel and Moser 1971, p 35). Comparing this form of the extended Hamiltonian

H_1 with those frequently found in the literature (Lanczos 1949, p 189, Synge 1960, p 143, Szebehely 1967, p 329, Stiefel and Scheifele 1971, Thirring 1977, Asorey *et al* 1983, Kuwabara 1984, Lichtenberg and Lieberman 1992, Wodnar 1995, Stump 1998, Tsiganov 2000, Struckmeier and Riedel 2002a), we note the additional scaling factor $k = \mathrm{d}t/\mathrm{d}s$. As will become clear in the context of extended canonical transformations, this factor is crucial to ensure the form-invariance of H_1 under non-trivial canonical time transformations. On the other hand, we observe that the scaling factor k must not be defined as an arbitrary differentiable function on T^*Q_1 in order to be compatible with Hamilton's variational principle. We will discuss this issue and its implications for the canonical transformation formalism in section 3.2.

As $\mathcal{H}$ stands for the value of H, the extended Hamiltonian H_1 of equation (6) occurs as the *implicit function* $H_1(\boldsymbol{q}_1, \boldsymbol{p}_1) = 0$. In view of the assertion that the extended Hamiltonian H_1 vanishes *identically* (Lanczos 1949, p 186), we observe that this is only true for the representation of H_1 on the $(2n + 1)$-dimensional presymplectic submanifold $T^*Q \times \mathbb{R}$, obtained by replacing $\mathcal{H}$ in (6) with the Hamiltonian function H. However, on the $(2n + 2)$-dimensional symplectic extended phase space T^*Q_1, the extended Hamiltonian H_1 does *not* vanish identically but constitutes the implicit constraint function $H_1(\boldsymbol{p}_1, \boldsymbol{q}_1) = 0$, which defines a $(2n + 1)$-dimensional hypersurface in T^*Q_1 on which the system's evolution takes place—analogously to the $(2n - 1)$-dimensional hypersurface in T^*Q that defines a regular energy surface through the constraint function $H(\boldsymbol{q}, \boldsymbol{p}) = H_0$ in the case of an autonomous Hamiltonian system.

We note furthermore that in the Lagrangian description on the extended tangent bundle TQ_1, reviewed in the appendix, the factor $\mathrm{d}t/\mathrm{d}s$ is the $(n + 1)$-th element of the extended tangent vector $\mathrm{d}\boldsymbol{q}_1/\mathrm{d}s \in T_{q_1}Q_1$, and hence appears as an independent variable in the argument list of the extended Lagrangian L_1, defined on TQ_1. In the Hamiltonian description on the extended cotangent bundle T^*Q_1, the scaling factor $k = \mathrm{d}t/\mathrm{d}s$ is no longer an independent function of the system evolution parameter, but takes on the role of a parameter function.

2.2. Canonical equations, Poisson brackets

With the canonical coordinate representation $\Lambda = \boldsymbol{p}_1\,\mathrm{d}\boldsymbol{q}_1$ of the extended 1-form Λ, we obtain the s-parametrization of the variational integral (2) as

$$\delta \int_{s_1}^{s_2} \left[\sum_{i=1}^{n+1} p_i(s) \frac{\mathrm{d}q^i(s)}{\mathrm{d}s} - H_1(\boldsymbol{q}_1(s), \boldsymbol{p}_1(s)) \right] \mathrm{d}s = 0. \tag{8}$$

This representation of Hamilton's variational principle for $H_1(\boldsymbol{q}_1, \boldsymbol{p}_1)$ formally agrees with the conventional description for a time-independent Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$. Therefore, the critical path within the extended phase space $(\boldsymbol{q}_1(s), \boldsymbol{p}_1(s)) \subset T^*Q_1$ is constituted by the solution of the extended set of canonical equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}s} = \frac{\partial H_1}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial H_1}{\partial q^i}, \qquad i = 1, \ldots, n + 1. \tag{9}$$

In contrast to the total time derivative of the original Hamiltonian H, the total s derivative of H_1 obviously vanishes identically by virtue of equations (9): $\mathrm{d}H_1/\mathrm{d}s \equiv 0$. Thus, the extended Hamiltonian $H_1(\boldsymbol{q}_1, \boldsymbol{p}_1)$ from equation (6) formally converts any given non-autonomous system $H(\boldsymbol{q}, \boldsymbol{p}, t)$ into an autonomous system in T^*Q_1. Inserting H_1 from (6), we may express the extended set of canonical equations (9) in terms of the conventional Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}, t)$ as

$$\frac{\mathrm{d}q^i}{\mathrm{d}s} = k \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -k \frac{\partial H}{\partial q^i}, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = k, \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}s} = k \frac{\partial H}{\partial t}. \tag{10}$$

The leftmost two equations are simply the conventional canonical equations with s the independent variable instead of t. This shows that the critical paths C_0 and $\tilde{C}_0$ from equations (1) and (2) are equivalent, as required. The rightmost equation from (10) states that the partial time derivative of H now constitutes a regular canonical equation—the equation of motion for $\mathcal{H}(s)$. Yet the conjugate equation of motion for $t(s)$ merely constitutes an *identity*. This reflects the fact that the variational principle of equation (2) does not provide additional information, compared to its formulation in equation (1). The parametrization of the time $t = t(s)$ thus remains undetermined. As a consequence, the critical path $\tilde{C}_0$ that is given as the solution of the extended set of canonical equations satisfies Hamilton's variational principle (2) for *all* differentiable parametrizations of time $t = t(s)$. According to (10), instants of s with $k = 0$ simply mean that the system 'freezes' at those points. A negative k describes a backward time flow as s increases. This is also a valid parametrization $t = t(s)$ as Hamilton's variational principle is invariant with respect to the time-reversal transformation $t \mapsto -t$.

The extended set of canonical equations formally (9) agrees with those derived by Lanczos (1949, p 189) and Synge (1960, p 144). Yet, inserting the ad hoc approach of an extended Hamiltonian $H_{\mathrm{LS}} = H - \mathcal{H}$ of Lanczos and Synge into (9) yields the canonical equation $\mathrm{d}t/\mathrm{d}s = -\partial H_{\mathrm{LS}}/\partial\mathcal{H} \equiv 1$ in place of the identity $\mathrm{d}t/\mathrm{d}s \equiv k$ in (10). The extended Hamiltonian H_{LS} thus restricts $t = t(s)$ to a fixed function of the system evolution parameter, s, and hence abolishes the idea of $t(s) \equiv q^{n+1}(s)$ as an ordinary canonical variable. It is now evident why the factor $k = \mathrm{d}t/\mathrm{d}s$ in the extended Hamiltonian H_1 from equation (6) is important. As we will see in the context of extended canonical transformations—which are associated with non-trivial time mappings $t(s) \mapsto t'(s)$—the *a priori* fixation of $t(s)$ is inadequate as $\mathrm{d}t/\mathrm{d}s \equiv 1$ and $\mathrm{d}t'/\mathrm{d}s \equiv 1$ cannot simultaneously hold true in the original and the transformed system. Therefore, an extended Hamiltonian $H_{\mathrm{LS}} = H - \mathcal{H}$ does not meet the requirement of conserving the form of the canonical equations under extended canonical transformations.

Obviously, the vector analysis operations on T^*Q_1 with its symplectic 2-form Ω from equation (7) are analogous to the corresponding operations on (T^*Q, ω). Let F denote a differentiable function on $F : T^*Q_1 \to \mathbb{R}$. We can associate with F a unique vector field X_F on T^*Q_1 by means of the interior product

$$X_F \lrcorner \Omega = -\mathrm{d}F.$$

In canonical coordinate description, this means that we have along the integral curves of X_F

$$\frac{\mathrm{d}q^i}{\mathrm{d}s} = \frac{\partial F}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial F}{\partial q^i}, \qquad i = 1, \ldots, n+1.$$

Identifying F with the extended Hamiltonian H_1, the related vector field that generates the system's dynamical evolution is then the extended Hamiltonian vector field X_{H_1}

$$X_{H_1} = k\left[\sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\right) + \frac{\partial H}{\partial t}\frac{\partial}{\partial \mathcal{H}} + \frac{\partial}{\partial t}\right], \tag{11}$$

satisfying

$$X_{H_1} \lrcorner \Omega = -k(\mathrm{d}H - \mathrm{d}\mathcal{H}) = -\mathrm{d}H_1.$$

Obviously, the $(2n + 1)$-dimensional submanifold of T^*Q_1, defined by $H_1 = 0$ of equation (6), is invariant under the flow of X_{H_1}. The restriction of X_{H_1} to the presymplectic manifold $T^*Q \times \mathbb{R}$ then yields the scaled representation $k\tilde{X}_H$ of the vector field $\tilde{X}_H$ (Abraham and Marsden 1976, p 376) that describes the dynamics of a time-dependent Hamiltonian system $(T^*Q \times \mathbb{R}, \omega_H, H)$.

Let F and G now denote two differentiable functions on T^*Q_1, with X_F and X_G the related dynamical vector fields. The extended symplectic 2-form induces a corresponding extended Poisson bracket $\{.,.\}_e$ via

$$-\Omega(X_F, X_G) \equiv \{F, G\}_e = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) - \frac{\partial F}{\partial t} \frac{\partial G}{\partial \mathcal{H}} + \frac{\partial F}{\partial \mathcal{H}} \frac{\partial G}{\partial t}. \tag{12}$$

As is easily verified, the fundamental Poisson brackets are

$$\{q^i, q^j\}_e = 0 \qquad \{p_i, p_j\}_e = 0 \qquad \{q^i, p_j\}_e = \delta^i_j \qquad i, j = 1, \ldots, n+1.$$

The extended set of canonical equations (9) yields for differentiable functions on T^*Q_1, hence functions F that do not explicitly depend on s

$$\{F, H_1\}_e = \frac{\mathrm{d}F}{\mathrm{d}s}.$$

The Poisson bracket representation of the canonical equations are thus obtained as

$$\{p_i, H_1\}_e = \frac{\mathrm{d}p_i}{\mathrm{d}s}, \qquad \{q^i, H_1\}_e = \frac{\mathrm{d}q^i}{\mathrm{d}s}, \qquad i = 1, \ldots, n+1.$$

This shows again that the canonical equations emerging from the Hamiltonian H_1 are just the conventional canonical equations, expressed in terms of the system evolution parameter s.

From equation (12), we easily confirm that the extended 2-form Ω is non-degenerate. Given two differentiable functions G on T^*Q_1, then $\{F, G\}_e = 0$ for all G obviously implies $F = 0$. Consequently, the extended Hamiltonian vector field X_{H_1} from equation (11) is uniquely determined by the extended Hamiltonian H_1. This establishes the complete analogy of our extended description of explicitly time-dependent Hamiltonian systems (T^*Q_1, Ω, H_1) with the conventional description of time-independent Hamiltonian systems (T^*Q, ω, H).

2.3. Canonical transformations, Liouville's theorem

By definition, the subset of diffeomorphisms $\phi : T^*Q_1 \to T^*Q_1$ that preserve the symplectic structure Ω are referred to as symplectic, or, synonymously, as canonical. This means that the induced map (pull-back) ϕ^* that acts on Ω must satisfy

$$\phi^*\Omega = \Omega. \tag{13}$$

As the closed canonical 2-form Ω is locally exact ($\Omega = \mathrm{d}\Lambda$), and the pull-back commutes with the exterior derivative, it can be concluded that $\phi^*\Lambda$ may differ from Λ at most by an exact 1-form. With a differentiable 'generating function' F_1, we thus obtain

$$\phi^*\Lambda - \Lambda + \mathrm{d}F_1 = 0.$$

The condition that the integrand of the extended variational principle (2) must remain form-invariant is then expressed in canonical coordinates for a function $F_1 : Q_1 \times Q_1 \times \mathbb{R} \to \mathbb{R}$ as

$$\sum_{i=1}^{n} p_i \,\mathrm{d}q^i - \mathcal{H}\,\mathrm{d}t - H_1\,\mathrm{d}s = \sum_{i=1}^{n} p_i' \,\mathrm{d}q'^i - \mathcal{H}'\,\mathrm{d}t' - H_1'\,\mathrm{d}s + \mathrm{d}F_1(\boldsymbol{q}, \boldsymbol{q}', t, t', s). \tag{14}$$

The requirement (14) thus automatically ensures the form-invariance of the canonical equations switching from the unprimed to the primed variables. Comparing the coordinates of the 1-form $\mathrm{d}F_1$

$$\mathrm{d}F_1 = \sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial q^i} \mathrm{d}q^i + \frac{\partial F_1}{\partial q'^i} \mathrm{d}q'^i \right) + \frac{\partial F_1}{\partial t} \mathrm{d}t + \frac{\partial F_1}{\partial t'} \mathrm{d}t' + \frac{\partial F_1}{\partial s} \mathrm{d}s$$

with (14), we obtain the transformation rules ($i = 1, \dots, n$)

$$p_i = \frac{\partial F_1}{\partial q^i}, \quad p'_i = -\frac{\partial F_1}{\partial q'^i}, \quad \mathcal{H} = -\frac{\partial F_1}{\partial t}, \quad \mathcal{H}' = \frac{\partial F_1}{\partial t'}, \quad H'_1 = H_1 + \frac{\partial F_1}{\partial s}.$$

We immediately conclude that the extended Hamiltonian H_1 is preserved if and only if the generating function F_1 does not explicitly depend on s

$$H'_1(\boldsymbol{q}', \boldsymbol{p}', t', \mathcal{H}') = H_1(\boldsymbol{q}, \boldsymbol{p}, t, \mathcal{H}) \quad \Longleftrightarrow \quad \partial F_1/\partial s = 0. \tag{15}$$

In order for the description in the new set of coordinates to be equivalent to the original set of unprimed coordinates, the transformation must be invertible. This is assured if and only if the Hessian condition

$$\det\left(\frac{\partial F_1}{\partial q^i \partial q'^j}\right) \neq 0 \tag{16}$$

of the $(n+1) \times (n+1)$ matrix of second partial derivatives of F_1 is satisfied along s.

With the help of the Legendre transformation

$$F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}', s) = F_1(\boldsymbol{q}, \boldsymbol{q}', t, t', s) + \sum_{i=1}^{n} q'^i p'_i - t'\mathcal{H}', \tag{17}$$

an equivalent generating function F_2 can be defined that depends on the original configuration space and the new momentum coordinates. If we compare the coefficients pertaining to the respective 1-forms $\mathrm{d}q^i$, $\mathrm{d}p'_i$, $\mathrm{d}t$, $\mathrm{d}\mathcal{H}'$ and $\mathrm{d}s$, we find the following coordinate transformation rules to apply for each index $i = 1, \dots, n$:

$$p_i = \frac{\partial F_2}{\partial q^i}, \quad q'^i = \frac{\partial F_2}{\partial p'_i}, \quad \mathcal{H} = -\frac{\partial F_2}{\partial t}, \quad t' = -\frac{\partial F_2}{\partial \mathcal{H}'}, \quad H'_1 = H_1 + \frac{\partial F_2}{\partial s}. \tag{18}$$

Equivalent transformation rules are induced by generating functions

$$\begin{aligned} &F_3(\boldsymbol{q}', \boldsymbol{p}, t', \mathcal{H}, s) = F_1(\boldsymbol{q}, \boldsymbol{q}', t, t', s) - \sum_{i=1}^{n} q^i p_i + t\mathcal{H} \\ &p'_i = -\frac{\partial F_3}{\partial q'^i}, \quad q^i = -\frac{\partial F_3}{\partial p_i}, \quad \mathcal{H}' = \frac{\partial F_3}{\partial t'}, \quad t = \frac{\partial F_3}{\partial \mathcal{H}}, \quad H'_1 = H_1 + \frac{\partial F_3}{\partial s}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &F_4(\boldsymbol{p}, \boldsymbol{p}', \mathcal{H}, \mathcal{H}', s) = F_3(\boldsymbol{q}', \boldsymbol{p}, t', \mathcal{H}, s) + \sum_{i=1}^{n} q'^i p'_i - t'\mathcal{H}' \\ &q^i = -\frac{\partial F_4}{\partial p_i}, \quad q'^i = \frac{\partial F_4}{\partial p'_i}, \quad t = \frac{\partial F_4}{\partial \mathcal{H}}, \quad t' = -\frac{\partial F_4}{\partial \mathcal{H}'}, \quad H'_1 = H_1 + \frac{\partial F_4}{\partial s}. \end{aligned}$$

The extended Hamiltonian H_1 is again preserved if the $F_{2,3,4}$ do not explicitly depend on s. Of course, the Hessian condition (16) must apply similarly for $F_{2,3,4}$ in order to assure the invertibility of the generated symplectic map.

With the set of extended canonical transformations providing a superset of the conventional ones, it is not astonishing that a conventional generating function $f_2(\boldsymbol{q}, \boldsymbol{p}', t)$ can always be reformulated as a particular extended generating function $F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}')$ by means of

$$F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}') = f_2(\boldsymbol{q}, \boldsymbol{p}', t) - t\mathcal{H}'. \tag{20}$$

The related transformation rules follow from equations (18) as

$$p_i = \frac{\partial f_2}{\partial q^i}, \quad q'^i = \frac{\partial f_2}{\partial p'_i}, \quad \mathcal{H} = \mathcal{H}' - \frac{\partial f_2}{\partial t}, \quad t' = t, \quad H'_1 = H_1.$$

As the generating function from equation (20) does not explicitly depend on s, the extended Hamiltonian H_1 is preserved, hence $(H' - \mathcal{H}')\,\mathrm{d}t'/\mathrm{d}s = (H - \mathcal{H})\,\mathrm{d}t/\mathrm{d}s$, which yields with the above coordinate transformation rules for t and $\mathcal{H}$

$$H' = H + \frac{\partial f_2}{\partial t}.$$

The conventional transformations—generated by f_2—thus coincide with the particular *subset* of general transformations generated by F_2 from equation (20). In other words, the conventional canonical transformations distinguish themselves by the fact that the system evolution parameter s can be replaced by the time t as the common independent variable of both the original and the destination systems. In this respect, the transformations generated by f_2 are the time-dependent canonical transformations on the presymplectic contact manifold $T^*Q \times \mathbb{R}$ of definition 5.2.6 of Abraham and Marsden (1978).

According to the fourth rule of equations (18), the 'extended' generating functions F_2 in general define non-trivial time transformations $t' \neq t$. As will become clear in the following example section, it is this *freedom* to relate a given system to a destination system at *different* instants of their respective time scales that enables us to formulate the Lorentz transformation as a particular canonical transformation in the extended phase space. Furthermore, we will show that only the generalized canonical transformation approach allows us to directly transform an explicitly time-dependent Hamiltonian system into a time-independent one.

The extended 2-form Ω has the highest non-vanishing power

$$\Omega^{n+1} = (n+1)! \cdot (-1)^{n(n+1)/2} \cdot \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_{n+1} \wedge \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^{n+1}.$$

The invariance of Ω with respect to canonical transformations thus induces the invariance of the extended volume form V_1

$$V_1 = \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_{n+1} \wedge \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^{n+1} = \mathrm{d}p'_1 \wedge \cdots \wedge \mathrm{d}p'_{n+1} \wedge \mathrm{d}q'^1 \wedge \cdots \wedge \mathrm{d}q'^{n+1}.$$

This is, in canonical coordinates, the extended phase-space formulation of Liouville's theorem.

It is obvious that an extended canonical transformation can only sustain the character of time t as a common parameter for all particles if the transformed time t' does not depend on the coordinates of different particles, i.e. if

$$\frac{\partial t'}{\partial q^i} = \frac{\partial t'}{\partial p_i} = 0. \tag{21}$$

Thus, the conditions from equation (21) impose restrictions on the functional dependence of extended generating functions in the case of multi-particle systems. We will encounter this restriction in the context of the Lorentz transformation, to be discussed in the following section.

Moreover, we easily convince ourselves that a 'space–time decomposition' $T^*Q_1 = T^*Q \times T^*\mathbb{R}$ of the symplectic extended phase space is preserved if in addition to (21)

$$\frac{\partial t'}{\partial \mathcal{H}} = \frac{\partial q'^i}{\partial \mathcal{H}} = \frac{\partial p'_i}{\partial \mathcal{H}} = 0. \tag{22}$$

With the conditions (21) and (22) fulfilled, the extended canonical transformation can be factorized as a conventional canonical transformation times a pure time scaling transformation of section 3.2. In contrast to an assertion of Asorey *et al* (1983), this is not necessarily the case: the Lorentz transformation, regarded as a particular canonical transformation in the extended phase space, does not satisfy all conditions, and hence cannot be factorized.

Furthermore, Liouville's theorem applies simultaneously in the subspace T^*Q, hence in the conventional phase space if

$$\frac{\partial t'}{\partial t}\frac{\partial \mathcal{H}'}{\partial \mathcal{H}} = 1 \tag{23}$$

holds in conjunction with equations (21) and (22).

3. Examples of canonical transformations in the extended phase space

3.1. Lorentz transformation

Here we consider two Cartesian frames of reference (x, y, z) and (x', y', z') that move with respect to each other at a constant velocity v. For simplicity, we first set up the coordinate system to be aligned so that the relative motion occurs along the x-axis. Under these circumstances, the y- and z-coordinates are not affected by the Lorentz transformation ($y' = y$, $z' = z$). As this transformation necessarily involves a non-trivial mapping of the respective time scales $t \mapsto t'$, it cannot be described in terms of a canonical transformation in the conventional phase space. Nevertheless, in the extended phase space, a generating function F_2 exists that exactly yields the Lorentz transformation rules,

$$F_2(x, p'_x, t, \mathcal{H}') = \gamma\left[p'_x(x - \beta ct) - \frac{\mathcal{H}'}{c}(ct - \beta x)\right]. \tag{24}$$

With c denoting the speed of light, the common notation is used to express the scaled relative velocity in the abbreviated form $\beta = v/c$. As usual, γ stands for the relativistic factor, defined by $\gamma^{-2} = 1 - \beta^2$.

For the particular generating function (24), the general rules for a canonical transformation in the extended phase space from equations (18) specialize to

$$\begin{aligned}
p_x &= \frac{\partial F_2}{\partial x} = \gamma\left(p'_x + \beta\frac{\mathcal{H}'}{c}\right), &\qquad \mathcal{H} &= -\frac{\partial F_2}{\partial t} = \gamma(\mathcal{H}' + \beta c p'_x),\\
x' &= \frac{\partial F_2}{\partial p'_x} = \gamma\,(x - \beta ct), &\qquad t' &= -\frac{\partial F_2}{\partial \mathcal{H}'} = \gamma\left(t - \frac{\beta}{c}x\right),\\
H'_1(x', p'_x, t', \mathcal{H}') &= H_1(x, p_x, t, \mathcal{H}), &\qquad H &= \gamma H' + \beta\gamma c p'_x.
\end{aligned}$$

As required, the transformation rule for the Hamiltonians H and H', derived from $H'_1 = H_1$, hence $(H - \mathcal{H})\,\mathrm{d}t = (H' - \mathcal{H}')\,\mathrm{d}t'$, agrees with the corresponding rule for their respective values, $\mathcal{H}$ and $\mathcal{H}'$. In complex notation, the coordinate transformation rules take on the familiar form of an orthogonal linear mapping

$$\begin{pmatrix} x' \\ \mathrm{i}ct' \\ p'_x \\ \mathrm{i}\mathcal{H}'/c \end{pmatrix} = \begin{pmatrix} \gamma & \mathrm{i}\beta\gamma & 0 & 0 \\ -\mathrm{i}\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & \gamma & \mathrm{i}\beta\gamma \\ 0 & 0 & -\mathrm{i}\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} x \\ \mathrm{i}ct \\ p_x \\ \mathrm{i}\mathcal{H}/c \end{pmatrix}. \tag{25}$$

We observe that the generating function (24) provides both the transformation rules for the (x, ct) coordinates—which commonly refer to the Lorentz transformation—and the related rules for the conjugate coordinates momentum and energy $(p_x, \mathcal{H}/c)$. This is not astonishing, as a canonical transformation always maintains the symplectic structure of the Hamiltonian in question—which requires the transformation rules for all canonical variables to be uniquely defined.

For the general case that both frames of reference are not aligned, their relative scaled velocity is expressed by the 3-component vector $\boldsymbol{\beta} = (\beta^i)$. With $\boldsymbol{q} = (x, y, z) = (q^i)$ and $\boldsymbol{p}' = (p'_x, p'_y, p'_z) = (p'_i)$, the general form of the generating function F_2 for the Lorentz transformation is then given by

$$F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}') = \gamma \frac{\mathcal{H}'}{c} \left[\sum_{i=1}^{3} \beta^i q^i - ct \right] + \sum_{i=1}^{3} p'_i \left[\sum_{k=1}^{3} \left(\delta^{ik} + (\gamma - 1) \frac{\beta^i \beta^k}{|\boldsymbol{\beta}|^2} \right) q^k - \gamma ct\beta^i \right] \tag{26}$$

which simplifies to the generating function (24) for the aligned case $\beta^1 = \beta_x = \beta$, $\beta^2 = \beta_y = 0$ and $\beta^3 = \beta_z = 0$.

The generating function (26) of the Lorentz transformation has the particular property

$$-\frac{\partial^2 F_2}{\partial q^i \partial \mathcal{H}'} = \frac{\partial t'}{\partial q^i} = -\frac{\gamma}{c} \beta^i, \tag{27}$$

which does not vanish for $\beta^i \neq 0$. This is the reason why it is impossible to maintain the meaning of the transformed time t' as a global parameter for the transformed coordinates p'_i and q'^i in multi-particle systems. Because of (27), the transformed time t' depends on q^i, which means that individual particles in the transformed system no longer carry the same time t'. Furthermore, a factorization of the symplectic extended phase space T^*Q_1 is not preserved as the conditions (21) and (22) are not satisfied. Hence, the Liouville volume form V_1 is preserved on T^*Q_1, but not the volume forms of any of its subspaces.

In order to relativistically describe multi-particle systems, Sorge *et al* (1989) introduced an $8N$-dimensional phase space, in which the positions and momenta of N particles are described as two 4-vectors that depend on a common evolution parameter. A more general, field-theoretical approach has been developed by Gotay *et al* (1997) on a 'multiphase space' that is endowed with a 'multisymplectic' $(n + 2)$-form as the covariant generalization of the symplectic 2-form of Hamiltonian mechanics.

We furthermore observe that the generating function (26) does not explicitly depend on s. According to (15), this means that extended Hamiltonians H_1 from (5) are always Lorentz-invariant—in contrast to non-extended Hamiltonians H. Of particular interest are, therefore, those non-extended Hamiltonians H that are form-invariant under the canonical transformation generated by F_2 from equation (26). As an example of how to convert a given non-Lorentz-invariant Hamiltonian H_{NL} into a Lorentz-invariant form, H_{L}, we consider the Hamiltonian of a particle with mass m and charge e within an electromagnetic field, defined by the potentials $(\boldsymbol{A}, \phi)$

$$H_{\mathrm{NL}}(\boldsymbol{q}, \boldsymbol{P}, t) = \frac{[\boldsymbol{P} - e\boldsymbol{A}(\boldsymbol{q}, t)]^2}{2m} + e\phi(\boldsymbol{q}, t). \tag{28}$$

As only expressions of the form $\boldsymbol{q}^2 - c^2t^2$ and $(\boldsymbol{P} - e\boldsymbol{A})^2 - (\mathcal{H} - e\phi)^2/c^2$ are invariant under the orthogonal transformation (25), the Hamiltonian (28) is obviously not Lorentz-invariant. In the extended phase space, however, a Lorentz-invariant form of (28) can easily be constructed by adding the required $\mathcal{H}$-term

$$H_{\mathrm{L}}(\boldsymbol{q}, \boldsymbol{P}, t, \mathcal{H}) = \frac{1}{2m} \left[(\boldsymbol{P} - e\boldsymbol{A})^2 - \frac{(\mathcal{H} - e\phi - mc^2)^2}{c^2} \right] + e\phi + mc^2. \tag{29}$$

We thus have chosen the usual normalization to define $H_{\mathrm{L}} = \mathcal{H} = mc^2$ for the particular case $\boldsymbol{P} = 0$ and zero field. The addition of the mc^2 terms merely describes a Lorentz-invariant shift of the origin. According to (4), the Hamiltonian (29) on T^*Q_1 is mapped into a Hamiltonian on the usual carrier manifold $T^*Q \times \mathbb{R}$ by replacing the Hamiltonian's value $\mathcal{H}$ with the Hamiltonian H_L itself,

$$H_\mathrm{L}(\boldsymbol{q}, \boldsymbol{P}, t) = \frac{1}{2m}\left[(\boldsymbol{P} - e\boldsymbol{A})^2 - \frac{(H_\mathrm{L} - e\phi - mc^2)^2}{c^2}\right] + e\phi + mc^2. \tag{30}$$

Solving (30) for H_L, we obtain the well-known result

$$H_\mathrm{L}(\boldsymbol{q}, \boldsymbol{P}, t) = \sqrt{c^2[\boldsymbol{P} - e\boldsymbol{A}(\boldsymbol{q}, t)]^2 + m^2c^4} + e\phi(\boldsymbol{q}, t),$$

i.e. the Lorentz-invariant form of the Hamiltonian H for a particle within an electromagnetic field.

3.2. Euler's time scaling transformation

In the context of the regularizing transformation of the three-body problem (Siegel and Moser 1971), we encounter from heuristic reasoning a replacement of the time t by a new independent variable, t', defined by

$$t'(t) = \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\xi(\tau)}.$$

We shall see in the following that this particular transformation constitutes a simple canonical transformation in the extended phase space. Namely, a canonical transformation that defines an identical mapping in $\boldsymbol{q}$ and $\boldsymbol{p}$ but merely 'scales' the extended variables $t(s)$ and $\mathcal{H}(s)$ is induced by the extended generating function

$$F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}') = \boldsymbol{q}\boldsymbol{p}' - \mathcal{H}' \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\xi(\tau)}, \tag{31}$$

with $\xi(t)$ denoting an *arbitrary* function of time only. The particular transformation rules emerge from the general rules (18) as

$$q'^i = q^i, \quad p'_i = p_i, \quad t' = \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\xi(\tau)}, \quad \mathcal{H}' = \xi(t)\mathcal{H}, \quad H'_1 = H_1. \tag{32}$$

From $H'_1(\boldsymbol{q}'_1, \boldsymbol{p}'_1) = H_1(\boldsymbol{q}_1, \boldsymbol{p}_1)$, hence from $H - \mathcal{H} = (H' - \mathcal{H}')\,\mathrm{d}t'/\mathrm{d}t$, we directly conclude

$$H'(\boldsymbol{q}(t'), \boldsymbol{p}(t'), t') = \xi(t) H(\boldsymbol{q}(t), \boldsymbol{p}(t), t).$$

Not surprisingly, the Hamiltonians H and H' follow again the same transformation rule as their respective values, $\mathcal{H}$ and $\mathcal{H}'$. Because of

$$\frac{\partial^2 F_2}{\partial q^i \partial \mathcal{H}'} = \frac{\partial^2 F_2}{\partial p'_i \partial \mathcal{H}'} = 0,$$

the transformed time t' retains the character of a global parameter that is common to all coordinates q'^i and p'_i in the transformed system. A factorization of the extended phase space $T^*Q_1 = T^*Q \times T^*\mathbb{R}$ is preserved, since additionally

$$\frac{\partial t'}{\partial \mathcal{H}} = \frac{\partial q'^i}{\partial \mathcal{H}} = \frac{\partial p'_i}{\partial \mathcal{H}} = 0.$$

Because furthermore

$$\frac{\partial t'}{\partial t}\frac{\partial \mathcal{H}'}{\partial \mathcal{H}} = 1,$$

the volume forms on the subspaces T^*Q and $T^*\mathbb{R}$ are separately conserved by means of the canonical transformation generated by (31), hence

$$V = \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n = V', \qquad \mathrm{d}t \wedge \mathrm{d}\mathcal{H} = \mathrm{d}t' \wedge \mathrm{d}\mathcal{H}'.$$

Following from the fact that $\xi(t)$ is an arbitrary function of time only, it can be freely identified with any combination of the canonical coordinates $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$, regarded as the *time functions* that are obtained from integrating the canonical equations. Of course, this does not mean that $\xi(t)$ acquires an explicit dependence on the coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$; hence an expression of $\xi(t)$ in terms of $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$ must not be inserted back into the Hamiltonian! Therefore, the identification of $\xi(t)$ with functions of the canonical coordinates $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$ is admissible only *after* all differentiations with respect to the canonical coordinates have been accomplished.

A simple example of how to formulate a time scaling transformation as an extended canonical transformation will be presented in the following for Euler's regularization of the Kepler equation of motion in one dimension. The Hamiltonian of this problem is given in normalized form by (Stiefel and Scheifele 1971)

$$H(x, p) = \frac{1}{2}p^2 - \frac{K^2}{x}, \qquad K^2 = \Gamma \cdot (M + m),$$

with Γ the gravitational constant, M, m the masses, and x the distance of the collision partners. As H does not explicitly depend on time, we have $\mathrm{d}\mathcal{H}/\mathrm{d}t = \partial H/\partial t = 0$, hence

$$\mathcal{H} = \frac{1}{2}p^2 - \frac{K^2}{x} = \text{const.} \tag{33}$$

The resulting equation of motion for x is obviously singular at the point of collision at $x = 0$,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{K^2}{x^2} = 0.$$

It was Euler who first worked out a transformation of the time scales that regularizes this equation of motion. The canonical transformation in the symplectic extended phase that defines this regularization transformation is generated by the function F_2 of equation (31) and the subsequent coordinate transformation rules (32). The transformed Hamiltonian H' is then

$$H'(x, p, t') = \xi(t')\left(\frac{1}{2}p^2 - \frac{K^2}{x}\right), \tag{34}$$

with the coordinates $x' = x$, $p' = p$ and ξ now understood as functions of t'. As a result of the fact that the transformation is canonical, the form of the canonical equations is preserved in the transformed system,

$$\frac{\mathrm{d}x}{\mathrm{d}t'} = \frac{\partial H'}{\partial p} = \xi(t')p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t'} = -\frac{\partial H'}{\partial x} = -\xi(t')\frac{K^2}{x^2}.$$

The equation of motion in terms of t' and the energy conservation relation from equation (33) follow as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t'^2} - \frac{1}{\xi}\frac{\mathrm{d}\xi}{\mathrm{d}t'}\frac{\mathrm{d}x}{\mathrm{d}t'} + \frac{K^2\xi^2}{x^2} = 0, \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}t'}\right)^2 = 2\xi^2\left(\mathcal{H} + \frac{K^2}{x}\right). \tag{35}$$

Having worked out the transformed equations of motion, we are now free to identify the as yet undetermined function $\xi(t')$ with an arbitrary function of the canonical variables $x(t')$ and $p(t')$, regarded as functions of time t', respectively. In doing so, we fix the correlation of the 'fictitious' time t' with the physical time t. In the present case we define

$$\xi(t') \equiv x(t') \qquad \Longrightarrow \qquad t(t') = \int_0^{t'} x(\tau)\,\mathrm{d}\tau.$$

The equation of motion and the energy conservation relation from equation (35) now reduce to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t'^2} - \frac{1}{x}\left(\frac{\mathrm{d}x}{\mathrm{d}t'}\right)^2 + K^2 = 0, \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}t'}\right)^2 = 2\mathcal{H}x^2 + 2K^2 x.$$

By finally inserting the energy conservation relation into the equation of motion, we obtain Euler's regularized equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t'^2} - 2\mathcal{H}x = K^2.$$

Summarizing, we can state that the time scaling transformation from equation (32) can be considered as a *finite* canonical transformation in the symplectic extended phase space. However, in order to maintain the consistency of the canonical transformation approach, the identification of the arbitrary time function $\xi(t')$ with a suitable combination of the *time functions* $\boldsymbol{q}(t')$ and $\boldsymbol{p}(t')$ can only be performed *after* having worked out the transformed canonical equations. In other words, the identification $\xi(t') \equiv x(t')$ must not be inserted back into the Hamiltonian H' of equation (34) since $\xi(t')$ continues to be a function of time only and does not 'acquire' an explicit dependence on the canonical variables. This contrasts with procedures sometimes are found in the literature (Cariñena and Ibort 1987, Cariñena *et al* 1988, Tsiganov 2000).

3.3. Time-dependent damped harmonic oscillator

The time-dependent harmonic oscillator model is frequently used as the first-order approximation for nonlinear, explicitly time-dependent Hamiltonian systems. We shall demonstrate in the following that a system of n particles that is confined within a time-dependent harmonic oscillator potential and that is subject to linear time-dependent damping forces can be mapped into a conventional undamped time-independent harmonic oscillator system by means of a single-canonical transformation in the extended phase space. The Hamiltonian of the original system is given by

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2}\,\mathrm{e}^{-F(t)}\boldsymbol{p}^2 + \tfrac{1}{2}\,\mathrm{e}^{F(t)}\omega^2(t)\boldsymbol{q}^2, \tag{36}$$

which yields the equations of motion

$$\ddot{q}^i + f(t)\dot{q}^i + \omega^2(t)q^i = 0, \qquad i = 1, \ldots, n, \quad f(t) = \dot{F}(t). \tag{37}$$

The destination Hamiltonian H'—with t' its independent variable—shall be the autonomous system

$$H'(\boldsymbol{q}', \boldsymbol{p}') = \tfrac{1}{2}\boldsymbol{p}'^2 + \tfrac{1}{2}\omega_0^2\boldsymbol{q}'^2. \tag{38}$$

A one-parameter family of functions $F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}')$ that generates the mapping of the Hamiltonian (36) into the Hamiltonian (38) has been found to be

$$F_2(\boldsymbol{q}, \boldsymbol{p}', t, \mathcal{H}') = \sqrt{\frac{\mathrm{e}^{F(t)}}{\xi(t)}}\boldsymbol{q}\boldsymbol{p}' + \frac{1}{4}\,\mathrm{e}^{F(t)}\left[\frac{\dot{\xi}(t)}{\xi(t)} - f(t)\right]\boldsymbol{q}^2 - \mathcal{H}'\int_0^t \frac{\mathrm{d}\tau}{\xi(\tau)}, \tag{39}$$

with the parameter $\xi(t)$, for the moment, an undetermined differentiable function of time. Because of the quadratic dependence on the canonical coordinates, the transformation rules (18) yield the linear mapping

$$\begin{pmatrix} q'^i \\ p'_i \end{pmatrix} = \begin{pmatrix} \sqrt{\mathrm{e}^{F(t)}/\xi(t)} & 0 \\ -\frac{1}{2}(\dot{\xi} - \xi f)\sqrt{\mathrm{e}^{F(t)}/\xi(t)} & \sqrt{\xi(t)/\mathrm{e}^{F(t)}} \end{pmatrix} \begin{pmatrix} q^i \\ p_i \end{pmatrix}. \tag{40}$$

The transformations of time t, energy $\mathcal{H}$, and extended Hamiltonian H_1 between both systems emerge from the generating function (39) as

$$\begin{aligned} t' &= \int_0^t \frac{\mathrm{d}\tau}{\xi(\tau)}, \\ \mathcal{H}' &= \xi\mathcal{H} - \frac{1}{2}(\dot{\xi} - \xi f)\boldsymbol{q}\boldsymbol{p} + \frac{1}{4}\,\mathrm{e}^{F(t)}(\ddot{\xi} - \dot{\xi} f - \xi\dot{f})\boldsymbol{q}^2, \\ H_1' &= H_1. \end{aligned} \tag{41}$$

We observe that the time-shift transformation between both systems is determined by the as yet unknown function $\xi(t)$. In any case, the transformed time t' does not depend on individual particles coordinates, hence retains the character of t as a global parameter for all particles. The correlation of the Hamiltonians H and H' follows from the transformation rule for H_1 of equation (41), hence from $(H - \mathcal{H})\,\mathrm{d}t/\mathrm{d}s = (H' - \mathcal{H}')\,\mathrm{d}t'/\mathrm{d}s$. With $\mathrm{d}t'/\mathrm{d}s = \xi^{-1}(t)\,\mathrm{d}t/\mathrm{d}s$ and inserting the Hamiltonian (36), we obtain H' as

$$H'(\boldsymbol{q}', \boldsymbol{p}', t') = \tfrac{1}{2}\boldsymbol{p}'^2 + \tfrac{1}{2}\boldsymbol{q}'^2\left[\tfrac{1}{2}\xi\ddot{\xi} - \tfrac{1}{4}\dot{\xi}^2 + \xi^2\left(\omega^2 - \tfrac{1}{2}\dot{f} - \tfrac{1}{4}f^2\right)\right],$$

having replaced all unprimed variables according to the rules (40) and (41). Hence, the destination Hamiltonian (38) indeed emerges if we identify

$$\tfrac{1}{2}\xi\ddot{\xi} - \tfrac{1}{4}\dot{\xi}^2 + \xi^2\left(\omega^2(t) - \tfrac{1}{2}\dot{f} - \tfrac{1}{4}f^2\right) = \omega_0^2 = \text{const.} \tag{42}$$

The primed system's potential is not time dependent if and only if $\mathrm{d}\omega_0^2\big/\mathrm{d}t' = 0$. This condition yields the linear third-order equation

$$\dddot{\xi}(t) + \dot{\xi}(4\omega^2(t) - 2\dot{f}(t) - f^2(t)) + \xi(4\omega\dot{\omega} - \ddot{f} - f\dot{f}) = 0. \tag{43}$$

As a result of this requirement, the function $\xi(t)$ is now determined—and hence the time correlation $t'(t)$ of both systems. It is precisely the extended canonical-transformation approach that enables us to properly adjust this time correlation. With $\xi(t)$ a solution of (43), the Hamiltonian H' does not depend on time explicitly. Expressed in the unprimed coordinates, the value $\mathcal{H}'$ of H' then yields an invariant of the original system (36)

$$\mathcal{H}' = \tfrac{1}{2}\,\mathrm{e}^{-F}\xi\boldsymbol{p}^2 - \tfrac{1}{2}(\dot{\xi} - \xi f)\boldsymbol{q}\boldsymbol{p} + \tfrac{1}{4}\,\mathrm{e}^{F(t)}(\ddot{\xi} - \dot{\xi}f - \xi\dot{f} + 2\xi\omega^2(t))\boldsymbol{q}^2 = \text{const.} \tag{44}$$

In terms of $\rho(t) = \sqrt{\xi(t)}$, the invariant (44) agrees with the invariant found by Leach (1978) for the case $n = 1$. Moreover, the invariant $\mathcal{H}'$ can be rendered a function of $\boldsymbol{q}$ and $\boldsymbol{p}$ only for this linear dynamical system. We easily verify that a particular solution $\xi(t)$ of equation (43) is given by

$$\xi(t) = \mathrm{e}^{F(t)}\boldsymbol{q}^2(t), \tag{45}$$

of course provided that $\boldsymbol{q}(t)$ is a solution vector of the equations of motion (37). Inserting (45) and its first and second time derivatives into (44), the invariant $\mathcal{H}'$ takes on the simple form

$$\mathcal{H}' = \boldsymbol{q}^2\boldsymbol{p}^2 - (\boldsymbol{q}\boldsymbol{p})^2 = \frac{1}{2}\sum_{i,j=1}^{n}\left(p_i q^j - p_j q^i\right)^2.$$

We immediately conclude that the individual sum terms

$$I_i^j = p_i q^j - p_j q^i, \qquad i, j = 1, \ldots, n$$

are also invariant. These quantities correspond to the conserved angular momenta in central force fields (Leach 1977). Consequently, the antisymmetric tensor $\left(I_i^j\right)$ is a non-trivial invariant of the n-particle time-dependent harmonic oscillator with time-dependent linear damping force (37).

The canonical equivalence of (36) and (38) is *physical* as long as the transformation rules (40) describe the correlation of *real* particles coordinates, hence as long as $\xi(t) > 0$. In order to show that $\xi(t)$ remains positive in the course of its time evolution if $\xi(0) > 0$, we make use of equation (42) to eliminate $\ddot{\xi}(t)$ in (44), which yields

$$2\mathcal{H}'\,\mathrm{e}^{-F(t)}\xi(t) = \omega_0^2\boldsymbol{q}^2 + \left[\xi\,\mathrm{e}^{-F}\boldsymbol{p} - \tfrac{1}{2}(\dot{\xi} - \xi f)\boldsymbol{q}\right]^2.$$

Since the right-hand side is obviously always positive, we immediately find $\xi(t) > 0$ for all t. Thus, for a time-dependent damped harmonic oscillator system (36) there always exists an equivalent genuine physical system of a time-independent undamped harmonic oscillator (38).

3.4. General time-dependent potential

As generalization of the previous example, we will now transform an n-degree-of-freedom Hamiltonian system with a general nonlinear time-dependent potential into a time-independent one. Let the Hamiltonian of the original system be given by

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \tfrac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{q}, t). \tag{46}$$

Again, we require a destination system H' of the same *form*, but with a potential V' that does *not* explicitly depend on the system's independent variable, t',

$$H'(\boldsymbol{q}', \boldsymbol{p}') = \tfrac{1}{2}\boldsymbol{p}'^2 + V'(\boldsymbol{q}'). \tag{47}$$

The most general 'extended' generating function F_2 that retains both the quadratic momentum dependence of H' and a momentum-independent potential V' turns out to be precisely the generating function (39) from the previous example (Struckmeier and Riedel 2002a 2002b), setting $F(t) \equiv 0$, and hence $f(t) \equiv 0$, as the actual system (46) does not include damping forces. The transformed Hamiltonian H' is then again obtained from the particular transformation rule from (41) for the extended Hamiltonians, $H_1' = H_1$, hence from $H' = \xi(H - \mathcal{H}) + \mathcal{H}'$. Inserting the Hamiltonian H of (46) and the transformation rule for $\mathcal{H}'$ of (41), and replacing the unprimed coordinates, we find

$$H'(\boldsymbol{q}', \boldsymbol{p}', t') = \tfrac{1}{2}\boldsymbol{p}'^2 + \tfrac{1}{4}\boldsymbol{q}'^2\big(\xi\ddot{\xi} - \tfrac{1}{2}\dot{\xi}^2\big) + \xi V(\sqrt{\xi}\boldsymbol{q}', t).$$

Thus, a Hamiltonian H' of the form of (47) turns out if the transformed potential V' is identfied with

$$V'(\boldsymbol{q}', t') = \tfrac{1}{4}\boldsymbol{q}'^2\left(\xi\ddot{\xi} - \tfrac{1}{2}\dot{\xi}^2\right) + \xi V(\sqrt{\xi}\boldsymbol{q}', t). \tag{48}$$

We can now make use of the freedom to appropriately adjust the time correlation $t'(t)$ between the original system (46) and the destination system (47) by requiring the new potential V' to be independent of its time t' explicitly

$$\frac{\partial V'}{\partial t'} \stackrel{!}{=} 0. \tag{49}$$

By this requirement, we now determine $\xi(t)$—which was initially defined in the generating function (39) as an arbitrary differentiable function of time. For the potential (48), the condition (49) evaluates to

$$\dddot{\xi}\boldsymbol{q}^2 + 4\dot{\xi}\left[V(\boldsymbol{q}, t) + \frac{1}{2}\boldsymbol{q}\frac{\partial V}{\partial \boldsymbol{q}}\right] + 4\xi\frac{\partial V}{\partial t} = 0. \tag{50}$$

The linear and homogeneous third-order differential equation (50) is equivalent to the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \dot{\xi}\\ \ddot{\xi}\end{pmatrix} = A(t)\begin{pmatrix}\xi\\ \dot{\xi}\\ \ddot{\xi}\end{pmatrix}, \qquad A(t) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -g_1(t) & -g_2(t) & 0\end{pmatrix} \tag{51}$$

with the coefficients g_1 and g_2 defined by

$$g_1(t) = \frac{4}{\boldsymbol{q}^2}\frac{\partial V}{\partial t}, \qquad g_2(t) = \frac{4}{\boldsymbol{q}^2}\left[V(\boldsymbol{q}, t) + \frac{1}{2}\boldsymbol{q}\frac{\partial V}{\partial \boldsymbol{q}}\right].$$

As by definition $\xi = \xi(t)$ embodies a function of t only, the coefficients g_1 and g_2 must also be functions of time only if the system (51) is to be solvable. This means that all spatial ($\boldsymbol{q}$-)dependencies in g_1 and g_2 must be conceived of as implicit time dependences via $\boldsymbol{q} = \boldsymbol{q}(t)$. In other words, the trajectory $\boldsymbol{q} = \boldsymbol{q}(t)$ as the solution of the equations of motion must be known in advance. Equation (51) should, therefore, be regarded as an *extension* of the system of canonical equations. In conjunction with the full set of canonical equations, the system (51) is closed and its functional dependence is uniquely determined.

Regarding the system matrix $A(t)$, we observe that its trace is *always* zero. Hence, the Wronski determinant of any 3×3 solution matrix $\Xi(t)$ of (51) is always constant, regardless of the particular form of the system's potential $V(\boldsymbol{q}, t)$. With the 3×3 unit matrix as a particular initial condition ($\Xi(0) = \mathbb{1}$), we thus obtain

$$\Xi(t) = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \dot{\xi}_1 & \dot{\xi}_2 & \dot{\xi}_3 \\ \ddot{\xi}_1 & \ddot{\xi}_2 & \ddot{\xi}_3 \end{pmatrix}, \qquad \Xi(0) = \mathbb{1}, \qquad \det \Xi(t) \equiv 1. \tag{52}$$

The transformation rule (41) now provides an integral of motion I for the original system (46) if and only if $\xi(t)$ and its time derivatives represent a linear combination of the three linearly independent vectors of the solution matrix $\Xi(t)$,

$$\mathcal{H}' = I = \xi(t)\mathcal{H} - \tfrac{1}{2}\dot{\xi}(t)\boldsymbol{q}\boldsymbol{p} + \tfrac{1}{4}\ddot{\xi}(t)\boldsymbol{q}^2 = \text{const}. \tag{53}$$

With the normalization $\Xi(0) = \mathbb{1}$, the three invariants, i.e. the three integration constants of the third-order system (51), can be written in matrix form in terms of the transpose solution matrix $\Xi^T(t)$,

$$\begin{pmatrix} \mathcal{H}_0 \\ -\frac{1}{2}\boldsymbol{q}_0\boldsymbol{p}_0 \\ \frac{1}{4}\boldsymbol{q}_0^2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \dot{\xi}_1 & \ddot{\xi}_1 \\ \xi_2 & \dot{\xi}_2 & \ddot{\xi}_2 \\ \xi_3 & \dot{\xi}_3 & \ddot{\xi}_3 \end{pmatrix} \begin{pmatrix} \mathcal{H} \\ -\frac{1}{2}\boldsymbol{q}\boldsymbol{p} \\ \frac{1}{4}\boldsymbol{q}^2 \end{pmatrix}_t . \tag{54}$$

The particular normalization $\Xi(0) = \mathbb{1}$ thus induces the invariants to represent the *initial values* of the Hamiltonian $\mathcal{H}$ and of the scalar products $\boldsymbol{q}\boldsymbol{p}$ and $\boldsymbol{q}^2$. One might have expected this result as the generic Hamiltonian system of equation (46) cannot have invariants other than its initial conditions and, trivially, combinations thereof. Nevertheless, what is actually surprising with equation (54) is the fact that the particular vector $\left(\mathcal{H}, -\frac{1}{2}\boldsymbol{q}\boldsymbol{p}, \frac{1}{4}\boldsymbol{q}^2\right)$ always depends *linearly* on its initial state, and that this mapping is associated with a *unit determinant*.

If the given system (46) is autonomous ($\partial V/\partial t \equiv 0$), then the linear equation (50) obviously has the particular solution $\xi_1(t) \equiv 1$. With regard to (54), this solution simply expresses the fact that the *value* of the Hamiltonian is a constant of motion ($\mathcal{H}(t) = \mathcal{H}_0$) if H does not depend on time explicitly. This well-known feature of autonomous Hamiltonian systems thus appears in our analysis in a more global context. Particularly, we observe that two other invariants always exist for autonomous systems that are associated with the non-constant solutions $\xi_2(t)$ and $\xi_3(t)$.

The physical meaning of equation (54) is expressed by the *time evolution* of the elements of the 'transfer matrix' $\Xi^T(t)$. As was shown by Struckmeier (2005), the properties of this map yield information with regard to the *regularity* of the system's time evolution.

4. Conclusions

With this present paper, we have provided a consistent reformulation of the classical Hamiltonian theory on the symplectic extended phase space. The extended description is based on a generalized understanding of Hamilton's variational principle by conceiving the time $t(s) = q^{n+1}(s)$ and the negative value $-\mathcal{H}(s) = p_{n+1}(s)$ of the Hamiltonian H as an additional pair of canonically conjugate variables that depends, like all other pairs of canonically conjugate variables, on a superordinated system evolution parameter s. With $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$ the canonical coordinate representation of the symplectic 2-form on T^*Q, the corresponding extended symplectic 2-form Ω on T^*Q_1 is then given by $\Omega = \omega - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t$. The extended 2-form Ω was shown to be non-degenerate. From Hamilton's variational principle, the general form of the extended Hamiltonian H_1 was derived, and its uniquely determined relation $H_1\,\mathrm{d}s = (H - \mathcal{H})\,\mathrm{d}t$ to the conventional Hamiltonian H was established. The result can now be summarized as follows:

> The symplectic Hamiltonian system (T^*Q_1, Ω, H_1), with $Q_1 = Q \times \mathbb{R}, \Omega = \omega - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t$, $H_1 = (H - \mathcal{H})\,\mathrm{d}t/\mathrm{d}s$, and H possibly time dependent is the proper canonical extension of the symplectic Hamiltonian system (T^*Q, ω, H) with time-independent Hamiltonian H.

Neither the frequently cited extended Hamiltonian $H_{\mathrm{LS}} = H - \mathcal{H}$ of Lanczos and Synge nor Cariñena's extended Hamiltonian $H_{\mathrm{C}} = f(H - \mathcal{H})$, with $f \in C^{\infty}(T^*Q_1)$, yield a formulation of dynamics on (T^*Q_1, Ω) that is analogous to that of (T^*Q, ω, H). In the first case, the subsequent canonical equation $\mathrm{d}t/\mathrm{d}s \equiv 1$ implies that H_{LS} is not preserved under non-trivial time transformations $t(s) \mapsto t'(s)$. In the second case, the obtained extended set of canonical equations cannot be derived from Hamilton's variational principle.

The canonically invariant form of the extended Hamiltonian H_1 that is consistent with Hamilton's variational principle turned out to coincide with the Hamiltonian of Poincaré's transformation of time. The well-known feature of Poincaré's approach to preserve the description of the system's dynamics was reflected by the fact that the extended set of canonical equations is equivalent to the conventional set of canonical equations.

In contrast, the formulation of extended canonical transformations on T^*Q_1 was shown to generalize the conventional presymplectic canonical transformation theory. Specifically, conventional canonical transformations were shown to constitute the particular subset of extended ones for which the system evolution parameter s can be replaced by the time t as a common independent variable of both the original and the destination system. With F denoting an extended generating function for an extended canonical transformation, we showed that the extended Hamiltonian H_1 is preserved if F does not explicitly depend on s. The extended Hamiltonian H_1 now meets the requirement to preserve the form of the canonical equations under extended canonical transformations generated by F.

We have furthermore worked out the restrictions that are to be imposed on extended generating functions in order for the transformed time t' to retain the meaning of t as a common parameter for all coordinates p'_i and q'^i. In a similar way, the conditions were obtained for Liouville's volume form to be separately conserved in the subspace T^*Q, i.e. in the conventional phase space.

In the first example, we demonstrated that the Lorentz transformation represents a particular canonical transformation in the symplectic extended phase space, which preserves H_1, for its generating function does not explicitly depend on s. The Lorentz transformation was shown to represent a particular extended canonical transformation that *cannot* be decomposed into a conventional canonical transformation times a canonical time scaling transformation. This canonical mapping clearly reveals the conditions for the non-extended Hamiltonian H to be also Lorentz-invariant. In demonstrating this in the case of a particle within an electromagnetic field, we obtain a guideline for converting non-Lorentz-invariant Hamiltonians H into Lorentz-invariant ones.

In the realm of celestial mechanics, the transformation of the physical time t to a 'fictitious' time t' is a long-established technique for regularizing singular equations of motion. With the theory of extended canonical transformations, we can now conceive regularization transformations of celestial mechanics as *finite* canonical transformations in the symplectic extended phase space that preserve H_1. This was demonstrated explicitly for Euler's regularization transformation of the one-dimensional Kepler motion.

Moreover, the generalized concept of canonical transformations permits a *direct* mapping of Hamiltonian systems with explicitly time-dependent potentials into time-independent Hamiltonian systems. An 'extended' generating function of type F_2 that defines a canonical mapping such as this was presented for both the time-dependent harmonic oscillator with time-dependent damping and for a general time-dependent potential. Similar to the regularization transformations, this generating function was defined to depend on an arbitrary time function $\xi(t)$. The freedom to finally commit oneself to a particular $\xi(t)$ was then utilized to render the destination system autonomous. The fundamental solution of the subsequent linear third-order differential equation for $\xi(t)$ was shown to provide information on the irregularity of the system's time evolution (Struckmeier 2005).

To conclude, the symplectic description of possibly time-dependent Hamiltonians H on the symplectic extended phase space (T^*Q_1, Ω) establishes a generalization of the usual presymplectic description on $(T^*Q \times \mathbb{R}, \omega_H)$. With the extended symplectic 2-form Ω, the induced extended Poisson bracket should then provide the means for a generalized Lie-algebraic description of dynamical systems with explicitly time-dependent Hamiltonians H on the symplectic extended phase space.

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Appendix. Extended Lagrangian description

A.1. Extended Euler–Lagrange equations

A time-independent Lagrangian L is defined as the mapping of the tangent bundle TQ into $\mathbb{R}$. If the Lagrangian L is explicitly time-dependent, then its domain is $TQ \times \mathbb{R}$, with $\mathbb{R}$ denoting the time axis. In local coordinates $(q^i, \dot{q}^i)$, the actual system path $(q^i(t), \dot{q}^i(t)) \subset TQ$ is given as the solution of the variational problem

$$\delta \int_{t_1}^{t_2} L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t)\, \mathrm{d}t \stackrel{!}{=} 0.$$

The variational integral can be expressed equivalently in parametric form if one replaces the time t as the independent variable with a new system evolution parameter, s. With

$$q^{n+1} = t, \qquad \dot{q}^i = \frac{\mathrm{d}q^i/\mathrm{d}s}{\mathrm{d}q^{n+1}/\mathrm{d}s},$$

we obtain (Lanczos 1949, Arnold 1989)

$$\delta \int_{s_1}^{s_2} L\left(q^1, \ldots, q^{n+1}, \frac{\mathrm{d}q^1/\mathrm{d}s}{\mathrm{d}q^{n+1}/\mathrm{d}s}, \ldots, \frac{\mathrm{d}q^n/\mathrm{d}s}{\mathrm{d}q^{n+1}/\mathrm{d}s}\right) \frac{\mathrm{d}q^{n+1}}{\mathrm{d}s}\, \mathrm{d}s \stackrel{!}{=} 0. \tag{A.1}$$

The integrand of (A.1) thus defines the extended Lagrangian $L_1 : TQ_1 \to \mathbb{R}$,

$$L_1\left(\boldsymbol{q}_1, \frac{\mathrm{d}\boldsymbol{q}_1}{\mathrm{d}s}\right) = L\left(q^1, \ldots, q^{n+1}, \frac{\mathrm{d}q^1/\mathrm{d}s}{\mathrm{d}q^{n+1}/\mathrm{d}s}, \ldots, \frac{\mathrm{d}q^n/\mathrm{d}s}{\mathrm{d}q^{n+1}/\mathrm{d}s}\right) \frac{\mathrm{d}q^{n+1}}{\mathrm{d}s}, \tag{A.2}$$

with $\boldsymbol{q}_1 = (\boldsymbol{q}, t) \in Q \times \mathbb{R} = Q_1$ the extended configuration space vector. The local coordinate representation of the actual system path $(q^i(s), \dot{q}^i(s)) \subset TQ_1$ is now given as the solution of the variational problem

$$\delta \int_{s_0}^{s_1} L_1\left(\boldsymbol{q}_1(s), \frac{\mathrm{d}\boldsymbol{q}_1(s)}{\mathrm{d}s}\right) \mathrm{d}s \stackrel{!}{=} 0. \tag{A.3}$$

As in the case of the conventional variational problem with a Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$, we find that (A.3) is globally fulfilled if the extended set of Euler–Lagrange equations is satisfied,

$$\frac{\partial L_1}{\partial \boldsymbol{q}_1} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L_1}{\partial(\mathrm{d}\boldsymbol{q}_1/\mathrm{d}s)}\right) = 0. \tag{A.4}$$

The following identities are readily derived from (A.2):

$$\frac{\partial L_1}{\partial \boldsymbol{q}} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\partial L}{\partial \boldsymbol{q}}, \qquad \frac{\partial L_1}{\partial(\mathrm{d}\boldsymbol{q}/\mathrm{d}s)} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \tag{A.5}$$

$$\frac{\partial L_1}{\partial t} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\partial L}{\partial t}, \qquad \frac{\partial L_1}{\partial(\mathrm{d}t/\mathrm{d}s)} = L - \dot{\boldsymbol{q}}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \tag{A.6}$$

which make it possible to rewrite (A.4) in terms of the conventional Lagrangian L

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s}\left[\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\right] = 0, \qquad \frac{\mathrm{d}\boldsymbol{q}(s)}{\mathrm{d}s}\left[\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\right] = 0. \tag{A.7}$$

We observe that both equations (A.7) are fulfilled if and only if the equations in brackets—the conventional Euler–Lagrange equations—are satisfied. Thus, the extended set of Euler–Lagrange equations (A.4) is equivalent to the conventional set and does *not* provide an additional equation of motion for $t = t(s)$. This result corresponds to the observation from (10) that the extended set of canonical equations does not furnish a substantial canonical equation for $\mathrm{d}t/\mathrm{d}s$, thus leaving the parametrization of time undetermined. Nevertheless, we may take advantage of having introduced the extended Lagrangian L_1: it is now possible to map L_1 by means of a Legendre transformation into an extended Hamiltonian H_1 whose domain is the symplectic manifold T^*Q_1.

A.2. Extended Hamiltonian H_1 as the Legendre transform of the extended Lagrangian L_1

Replacing all derivatives $\mathrm{d}q^i/\mathrm{d}s$ with $c\,\mathrm{d}q^i/\mathrm{d}s$, $c \in \mathbb{R}$, we realize that L_1 from equation (A.2) is a homogeneous form of first order in the $n + 1$ variables $\mathrm{d}q^1/\mathrm{d}s, \ldots, \mathrm{d}q^{n+1}/\mathrm{d}s$. Hence, Euler's theorem on homogeneous functions yields the identity (Lanczos 1949)

$$\sum_{i=1}^{n+1} \frac{\partial L_1}{\partial(\mathrm{d}q^i/\mathrm{d}s)}\frac{\mathrm{d}q^i}{\mathrm{d}s} \equiv L_1. \tag{A.8}$$

For the indices $i = 1, \ldots, n$, the partial derivatives of L_1 along the fibres $\mathrm{d}q^i/\mathrm{d}s$ define the generalized canonical momenta p_i,

$$\frac{\partial L_1}{\partial(\mathrm{d}q^i/\mathrm{d}s)} \equiv \frac{\partial L}{\partial \dot{q}^i} \equiv p_i, \qquad i = 1, \ldots, n. \tag{A.9}$$

The partial derivative of L_1 with respect to $\mathrm{d}q^{n+1}/\mathrm{d}s$ follows from its definition in (A.2) as

$$\frac{\partial L_1}{\partial(\mathrm{d}q^{n+1}/\mathrm{d}s)} \equiv L - \sum_{i=1}^{n} p_i \dot{q}^i \equiv -H(\boldsymbol{q}, \boldsymbol{p}, t). \tag{A.10}$$

Inserting (A.9) and (A.10) into the identity (A.8), the extended Lagrangian L_1 takes on the form

$$L_1 \equiv \sum_{i=1}^{n} p_i \frac{\mathrm{d}q^i}{\mathrm{d}s} - H(\boldsymbol{q}, \boldsymbol{p}, t)\frac{\mathrm{d}q^{n+1}}{\mathrm{d}s}. \tag{A.11}$$

With the extended Hamiltonian H_1 as the Legendre transform of L_1

$$H_1 \equiv \sum_{i=1}^{n+1} p_i \frac{\mathrm{d}q^i}{\mathrm{d}s} - L_1,$$

we find, inserting the identity for L_1 from (A.11), that the index $n + 1$ furnishes the only remaining term

$$H_1 \equiv [H(\boldsymbol{q}, \boldsymbol{p}, t) + p_{n+1}] \frac{\mathrm{d}q^{n+1}}{\mathrm{d}s}. \tag{A.12}$$

From equation (A.10), we conclude that in the description of the extended phase space T^*Q_1 the canonical variable $p_{n+1}(s) \in \mathbb{R}$, i.e. the derivative of L_1 along the fibre $\mathrm{d}t/\mathrm{d}s$, is *uniquely determined* by the negative value $-\mathcal{H}(s)$ of the Hamiltonian $H : T^*Q \times \mathbb{R} \to \mathbb{R}$

$$p_{n+1}(s) \equiv -\mathcal{H}(s) \stackrel{\not\equiv}{=} -H(\boldsymbol{q}(s), \boldsymbol{p}(s), t(s)). \tag{A.13}$$

With $p_{n+1} = -\mathcal{H}$ and $q^{n+1} = t$, the extended Hamiltonian H_1 from (A.12) is finally obtained as

$$H_1(\boldsymbol{q}, \boldsymbol{p}, t, \mathcal{H}) \equiv [H(\boldsymbol{q}(s), \boldsymbol{p}(s), t(s)) - \mathcal{H}(s)] \frac{\mathrm{d}t(s)}{\mathrm{d}s} \stackrel{\not\equiv}{=} 0. \tag{A.14}$$

The extended Hamiltonian (A.14) coincides with the Hamiltonian H_1 previously obtained in (6). As the extended Hamiltonian H_1 does not vanish *identically* in T^*Q_1, the partial derivatives of (A.14) are nonzero in general. Therefore, in contrast to the assertion of Lanczos (1949, p 187), the extended Hamiltonian H_1 must not be eliminated from the integrand of the generalized variational problem (8).

References

Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (Boulder, CO: Westview Press)
Arnold V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (New York: Springer)
Asorey M, Cariñena J F and Ibort L A 1983 *J. Math. Phys.* **24** 2745
Cariñena J F and Ibort L A 1987 *Nuovo Cimento* **98** 172
Cariñena J F, Ibort L A and Lacomba E A 1988 *Celes. Mech.* **42** 201–13
Frankel Th 2001 *The Geometry of Physics* (Cambridge: Cambridge University Press)
Gotay M J 1982 *Proc. Am. Math. Soc.* **84** 111
Gotay M J, Isenberg J, Marsden J E and Montgomery R 1997 *Momentum Maps and Classical Relativistic Fields: Part I. Covariant Field Theory* online at http://www.math.hawaii.edu/~gotay/GIMMsyI.pdf
José J V and Saletan E J 1998 *Classical Dynamics* (Cambridge: Cambridge University Press)
Kuwabara R 1984 *Rep. Math. Phys.* **19** 27–38
Lanczos C 1949 *The Variational Principles of Mechanics* (Toronto, Ontario: University of Toronto Press) Reprint 1986 4th edn (New York: Dover Publications)
Leach P G L 1977 *J. Math. Phys.* **18** 1608
Leach P G L 1978 *SIAM J. Appl. Math.* **34** 496
Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Motion* (New York: Springer)
Marsden J E and Ratiu T S 1999 *Introduction to Mechanics and Symmetry* (New York: Springer)
Siegel C L and Moser J K 1971 *Lectures on Celestial Mechanics* (Berlin: Springer)
Sorge H, Stöcker H and Greiner W 1989 *Nucl. Phys.* A **498** 567c
Stiefel E L and Scheifele G 1971 *Linear and Regular Celestial Mechanics* (Berlin: Springer)
Struckmeier J and Riedel C 2001 *Phys. Rev.* E **64** 026503
Struckmeier J and Riedel C 2002a *Ann. Phys. Lpz.* **11** 15–38
Struckmeier J and Riedel C 2002b *Phys. Rev.* E **66** 066605
Struckmeier J 2005 'Energy-second-moment map' analysis as an approach to estimating the irregularity of Hamiltonian systems (to be published)
Stump D R 1998 *J. Math. Phys.* **39** 3661
Synge J L 1960 *Encyclopedia of Physics* vol 3/1 ed S Flügge (Berlin: Springer)
Szebehely V 1967 *Theory of Orbits* (New York: Academic Press)
Thirring W 1977 *Lehrbuch der Mathematischen Physik* (Wien: Springer)
Tsiganov A V 2000 *J. Phys. A: Math. Gen.* **33** 4169–82
Wodnar K 1995 Symplectic mappings and Hamiltonian systems *Perturbation Theory and Chaos in Nonlinear Dynamics With Emphasis to Celestial Mechanics* ed J Hagel, M Cunha and R Dvorak (Portugal: Universidade da Madeira, Funchal)